Last Time: Diagonalization of matrices.

Algorithm: Let M be square matrix.

① Characteristic Poly $p_M(\lambda) = \det(M - \lambda I)$

② Solve $p_M(\lambda)$ for eigenvalues.

③ Build a basis for $\mathbb{R}^n$ (or $\mathbb{C}^n$) of eigenvalues (Eigenbasis).

$\hookrightarrow$ Compute bases of each Eigenspace.

④ Supposing each eigenvalue λ has geom mult = alg mult, the result of these computations is a basis E.

⑤ Realize $M = PDP^{-1}$ where $P = [E] = \mathrm{Rep}_{E,\mathcal{E}_n}(id)$, and $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ is the matrix of eigenvalues.

Ex: We diagonalize $M = \begin{bmatrix} -9 & -4 \\ 24 & 11 \end{bmatrix}$.

If M is $n\times n$ and M has n distinct e-values, then M is diag'ble.

Char Poly: $p_M(\lambda) = \det(M - \lambda I) = \det\begin{bmatrix} -9-\lambda & -4 \\ 24 & 11-\lambda \end{bmatrix}$

$$= (-9-\lambda)(11-\lambda) - 24(-4)$$
$$= -99 - 2\lambda + \lambda^2 + 96$$
$$= \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$$
$$= (3-\lambda)(-1-\lambda)$$

$\therefore$ We have eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$

(NB: Have 2 distinct e-values for this 2×2 matrix, So M is automatically diagonalizable !!).

$\lambda_1 = 3$: $V_{\lambda_1} = \mathrm{null}(M - \lambda_1 I) = \mathrm{null}\begin{bmatrix} -9-3 & -4 \\ 24 & 11-3 \end{bmatrix} = \mathrm{null}\begin{bmatrix} -12 & -4 \\ 24 & 8 \end{bmatrix}$

$$= \mathrm{null}\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} = \mathrm{null}\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} \in V_{\lambda_1}$ iff $3x + y = 0$ iff $y = -3x$

iff $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -3x \end{bmatrix} = x\begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$\therefore B_{\lambda_1} = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right\}$ is a basis of V_{λ_1}.

$\underline{\lambda_2 = -1}$: $V_{\lambda_2} = \text{null}(M - \lambda_2 I) = \text{null}\begin{bmatrix} -9+1 & -4 \\ 24 & 11+1 \end{bmatrix} = \text{null}\begin{bmatrix} -8 & -4 \\ 24 & 12 \end{bmatrix}$

$$= \text{null}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} = \text{null}\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} \in V_{\lambda_2}$ iff $2x + y = 0$ iff $y = -2x$

iff $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -2x \end{bmatrix} = x\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

$\therefore B_{\lambda_2} = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$ is a basis of V_{λ_2}

<u>Eigenbasis:</u> Let $E = B_{\lambda_1} \cup B_{\lambda_2} = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$. Now let

$P = \text{Rep}_{E, \mathcal{E}_2}(id) = \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix}$, and $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$.

<u>Check:</u> We'll verify $M = PDP^{-1}$: $\left(P^{-1} = \dfrac{1}{\underbrace{-2-(-3)}_{=1}} \begin{bmatrix} -2 & -1 \\ 3 & 1 \end{bmatrix} \right)$

$$PDP^{-1} = \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} -6 & -3 \\ -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & -4 \\ 24 & 11 \end{bmatrix} = M$$

$\square$

<u>Ex:</u> Let $M = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$. Diagonalize if possible.

<u>Sol:</u> <u>Char poly:</u> $p_M(\lambda) = \det(M - \lambda I) = \det\begin{bmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 + 1$

$\therefore$ <u>eigenvalues:</u> $p_M(\lambda) = 0 \iff (2-\lambda)^2 = -1 \iff 2 - \lambda = \pm i$

$\iff \lambda = 2 \pm i$ ← complex e-vals.

$\therefore$ M diag'ble over $\mathbb{C}$.

(<u>NB:</u> Not every diag'ble matrix is diag'ble over $\mathbb{R}$...)

$\underline{\lambda_1 = 2 + i}$: $V_{\lambda_1} = \text{null}(M - \lambda_1 I) = \text{null}\begin{bmatrix} 2-(2+i) & 1 \\ -1 & 2-(2+i) \end{bmatrix}$

$$= \text{null}\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} = \text{null}\begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} \in V_{\lambda_1}$ iff $x + iy = 0$ iff $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -iy \\ y \end{bmatrix} = y\begin{bmatrix} -i \\ 1 \end{bmatrix}$. $\therefore B_{\lambda_1} = \left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$.

$\underline{\lambda_2 = 2-i}$: $V_{\lambda_2} = \text{null}(M - \lambda_2 I) = \text{null}\begin{bmatrix} 2-(2-i) & 1 \\ -1 & 2-(2-i) \end{bmatrix} = \text{null}\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$

$$= \text{null}\begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} \in V_{\lambda_2}$ iff $x - iy = 0$ iff $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} iy \\ y \end{bmatrix} = y\begin{bmatrix} i \\ 1 \end{bmatrix}$. $\therefore B_{\lambda_2} = \left\{\begin{bmatrix} i \\ 1 \end{bmatrix}\right\}$.

$\underline{\text{Eigenbasis}}$: $E = B_{\lambda_1} \cup B_{\lambda_2} = \left\{\begin{bmatrix} -i \\ 1 \end{bmatrix}, \begin{bmatrix} i \\ 1 \end{bmatrix}\right\}$ is a basis of $\mathbb{C}^2$.

$\therefore M = PDP^{-1}$ for $P = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix}$.

$\underline{\text{Check}}$: $\underline{PDP^{-1}} = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix} \cdot \frac{1}{2}i\begin{bmatrix} 1 & -i \\ -1 & -i \end{bmatrix}$ $\quad \left(P^{-1} = \frac{1}{-i-i}\begin{bmatrix} 1 & -i \\ -1 & -i \end{bmatrix}\right)$

$$\frac{1}{-i} = \frac{i}{-i \cdot i} = \frac{i}{-(-1)} = i$$

$$= \frac{1}{2}i\begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2+i & -2i+1 \\ -2+i & -2i-1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ i & i \end{bmatrix}\begin{bmatrix} 2+i & 1-2i \\ -2+i & -1-2i \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2+i+2-i & 1-2i+1+2i \\ 2i-1-2i-1 & i+2-i+2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4 & 2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \underline{M}$$

$\left(\underline{\text{NB}}: \; M = PDP^{-1} \iff P^{-1}M = DP^{-1} \iff P^{-1}MP = D\right)$ $\boxtimes$

$\underline{\text{Ex}}$: Diagonalize $M = \begin{bmatrix} -4 & 1 \\ -1 & -6 \end{bmatrix}$ if possible.

$\underline{\text{Sol}}$: $p_M(\lambda) = \det(M - \lambda I) = \det\begin{bmatrix} -4-\lambda & 1 \\ -1 & -6-\lambda \end{bmatrix} = (4+\lambda)(6+\lambda)+1$

$$= (24 + 10\lambda + \lambda^2) + 1 = \lambda^2 + 10\lambda + 25 = (\lambda+5)^2 = \boxed{(-5-\lambda)^2}$$

$\therefore \underline{\lambda_1 = -5}$: $V_{\lambda_1} = \text{null}\begin{bmatrix} -4+5 & 1 \\ -1 & -6+5 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} \in V_{\lambda_1}$ iff $x + y = 0$ iff $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -x \end{bmatrix} = x\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore B_{\lambda_1} = \left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ is a basis of V_{λ_1}.

$\underline{\text{NB}}$: Geometric mult of λ_1 is $1 < 2$ and 2 is the alg mult of λ_1, so M is not Diag'able over $\mathbb{C}$. $\boxtimes$

(**Comment:** If M has exactly 1 e-value, it diagonalizes iff it was already diagonal.

$$PDP^{-1} = P(\lambda I)P^{-1} = \lambda(PIP^{-1}) = \lambda(PP^{-1}) = \lambda I = D$$

when D has a unique eigenvalue λ.)

Ex: Diagonalize $M = \begin{bmatrix} -5 & 0 & 6 \\ -3 & 1 & 3 \\ -3 & 0 & 4 \end{bmatrix}$ if possible.

Sol: $p_M(\lambda) = \det(M - \lambda I) = \det\begin{bmatrix} -5-\lambda & 0 & 6 \\ -3 & 1-\lambda & 3 \\ -3 & 0 & 4-\lambda \end{bmatrix}$

$$= -0 + (1-\lambda)\det\begin{bmatrix} -5-\lambda & 6 \\ -3 & 4-\lambda \end{bmatrix} - 0$$

$$= (1-\lambda)\big((-5-\lambda)(4-\lambda) - (-3)6\big)$$

$$= (1-\lambda)(-20 + \lambda + \lambda^2 + 18)$$

$$= (1-\lambda)(\lambda^2 + \lambda - 2) = (1-\lambda)(\lambda - 1)(\lambda + 2)$$

$$= (1-\lambda)^2(-2-\lambda)^1$$

$\therefore$ have $\lambda_1 = 1$ an e-value of alg mult 2 and $\lambda_2 = -2$ an e-value of alg mult 1.

$\underline{\lambda_1 = 1}$: $V_{\lambda_1} = \text{null}(M - \lambda_1 I) = \text{null}\begin{bmatrix} -5-1 & 0 & 6 \\ -3 & 1-1 & 3 \\ -3 & 0 & 4-1 \end{bmatrix} = \text{null}\begin{bmatrix} -6 & 0 & 6 \\ -3 & 0 & 3 \\ -3 & 0 & 3 \end{bmatrix}$

$$= \text{null}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V_{\lambda_1}$ iff $x - z = 0$ iff $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ x \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\therefore B_{\lambda_1} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis of V_{λ_1}.

(so λ_1 has geometric mult equal to it's alg mult !!).

$\underline{\lambda_2 = -2}$: $V_{\lambda_2} = \text{null}(M - \lambda_2 I) = \text{null}\begin{bmatrix} -5+2 & 0 & 6 \\ -3 & 1+2 & 3 \\ -3 & 0 & 4+2 \end{bmatrix} = \text{null}\begin{bmatrix} -3 & 0 & 6 \\ -3 & 3 & 3 \\ -3 & 0 & 6 \end{bmatrix}$

$$= \text{null}\begin{bmatrix} 1 & 0 & -2 \\ -3 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore\ V_{\lambda_2} \ni \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ iff $\begin{cases} x \quad - 2z = 0 \\ \quad y - z = 0 \end{cases}$ iff $\begin{cases} x = 2t \\ y = t \\ z = t \end{cases}$ iff $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

$\therefore\ B_{\lambda_2} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis of V_{λ_2}.

Eigenbasis: $E = B_{\lambda_1} \cup B_{\lambda_2} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$,

so M diagonalizes over $\mathbb{R}$.

Indeed $M = PDP^{-1}$ for $P = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

P^{-1} computation: $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{c|ccc} I & -1 & 0 & 2 \\ & -1 & 1 & 1 \\ & 1 & 0 & -1 \end{array}\right]$

$\therefore\ P^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$.

Check: $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \stackrel{?}{=} M$